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A symbolic manipulator to introduce pupils to algebra theory

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Abstract

Within the theoretical framework of Vygotsky's theory, the paper presents a new algebra software. The main features of the software are described and their potentialities are discussed. According to the Vygotskian theory, expressions and commands may be thought as external signs of the Algebraic theory, and as such, they may become instruments of semiotic mediation: In other terms they can be used by the teacher, in the concrete realisation of classroom activity, according to the motive of introducing pupils to syntactic manipulation as a theoretical activity.

1 Introduction

As clearly shown by previous research Studies the evolution of algebraic symbolism can be described in terms of "procedural-structural" terms, and, in particular, according to the psychological model described by Sfard (1991) this evolution requires a long period of transition.

Many research Studies show that the procedural character of pupils' conceptions related to literal terms and expressions tends to persist; at the same time, although symbolic manipulations of literal expressions is largely present in school practice, the absence of "structural conceptions" appears evident (Kieran, 1992, p. 397).

In the Italian school, pupils begin fairly soon to be trained in simplifying expressions (first numerical and then literal) and this training is intensively practiced at grade 9, when the first months of the school year are devoted to pupils introduction to 'Algebra'.

Limits related to a procedural approach to symbolic manipulation have been often pointed out, so as the need of a "structural-relational" approach in order to master symbolic manipulation in a productive way (Arzarello, 1991).

Poor strategic decisions has been described, made by students with extensive algebra experience, but unable to identify the right transformation to be accomplished: when the task does not explicitly indicate what transformation has to be performed, pupils are unable to take a decision, "go around in circles" (Kieran, 1992, p. 397) carrying out transformations without any clear gaol.

2 Algebra as a theoretic system

A key point of structural approach is the notion of "equivalence relation" between expressions. Actually, expressions manipulation means substituting an expression with another one which is equivalent. The meaning of the words "*expression*" and "*equivalent*" are not univocally, and a priori, determined, but it shell be so, once a set of axioms is accepted.

Let's consider a very generic example, let G be the group spanned by the elements **{1,a,b,c,A,B,C}** with the operator "•", where **A**, **B**, and **C** are the inverse elements of **a**, **b** and **c**, and **1** is the neutral element. In this example an algebraic expression can be defined as follows:

- If **x** belongs to **G**, then **x** is an algebraic expression;
- If **x** and **y** are algebraic expressions then **x•y** is an algebraic expression.

Following this definition we have that **a•(b•c)** and **(c•b)((a•c)•(b•b))** are both algebraic expressions[1]. Now the group axioms (**G** is a group) give a meaning to the statement *equivalent expressions*. In particular the associative property of the group **G** tells us that the expressions **a•(b•c)** and **(a•b)•c** are equivalent, furthermore, the expression **(c•b)((a•c)•(b•b))** is equivalent to the expression **(c•b)•(((a•c)•b)•b))**. If we also chose **G** to be commutative group, then **a•(b•c)** is equivalent to **(b•c)•a**, while if the group is not commutative this equivalence might not hold.

In conclusion, we consider *"symbolic manipulation"* as characterised by activities of transformation of expressions using the rules given by the assumed axioms and definitions[2]. Thus, symbolic manipulation makes sense within a theoretic system. Certainly this perspective is not very common in school practice (at least in Italy), yet it is exactly the perspective we assumed.

A previous study project, concerning pupils introduction to geometry theory (Mariotti et al., 1997, 2000), clearly showed how a computer environment may offer a support to overcome the well known difficulties related to theoretical perspective. In particular Mariotti analysed the semiotic mediation that can be accomplished by the teacher using specific instruments offered by the Cabri environment (Mariotti, in press) .

In the same stream a research project, still in progress, has been set up; a computer microworld, L'Algebrista (Cerulli 1999, Cerulli&Mariotti, 2000), was designed incorporating the axioms

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[1] From now on we are going to use the word *expression* to indicate *algebraic expression*.

[2] An example of definition which may be used as an instrument to transform expressions is the definition of power; for instance this definition allows to substitute **(a+1)•(a+1)•(a+1)** with $\mathbf{(a+1)^3}$ and vice versa.

defining the algebraic equivalence relation. A prototype was realised and experimented in ninth grade classes.

Differently from other symbolic manipulators, L'Algebrista is based on *elementary transformations*, that is transformations corresponding to applying a single axiom (or theorem or definition) at each step, with no automatic iteration.

Our basic hypothesis is that algebra's axioms, definitions and theorems, are the main elements involved in the transformation of expressions. In our microworld particular signs refer to such entities as expressions, axioms …; the manipulation of such signs corresponds to symbolic manipulation. In other terms, in the microworld a physical counterpart of expressions and axioms allows the user to visualise and make explicit the mathematical entities end relationships which are involved in symbolic manipulation.

Algebra theory, as far as imbedded in the microworld, is evoked by the expressions and the commands available in the L'Algebrista. According to the Vygotskian theory (Mariotti, in press), expressions and commands may be thought as external signs of the Algebraic theory, and as such, they may become instruments of semiotic mediation (Vygotsky , 1978). In other terms they can be used by the teacher, in the concrete realisation of classroom activity, according to the motive of introducing pupils to syntactic manipulation as a theoretical activity.

3 L'Algebrista: an overview

The software is a microworld incorporating the basic theory of algebraic expressions. Activities in the microworld consist in transforming expressions, a chain of such transformations correspond to the proof of the equivalence of expressions.

What follows is a list of the main ideas underling L'Algebrista:

- A **symbolic manipulator** which is totally under the user's control. It is going to be a microworld of algebraic expressions where the user can transform expressions on the basis of the fundamental properties of operations, which stand for the axioms of the local theory.
- Axioms are represented by the "buttons of the properties of the operations" which must not have any implicit behaviour; the buttons must realise only transformations which are **directly** implied by the axiom they represent. Furthermore, a button must not apply recursively an axiom, but only once.
- Buttons that represent equivalence relationships must be reversible and must include the inverse functionality as well. This is required to make explicit the meaning of equivalence

between expressions, and to associate the correct meaning of equivalence to the "equal" sign ("=").

- Some buttons will represent or recall conventions of the mathematical community, while other will represent or recall conventions, negotiated within the classroom community. Thus there will be a chance to make explicit some conventional aspects of the activities, in order to make explicit the conventionality of Mathematics.
- The interaction is based on direct manipulation, using the mouse to select expressions and to click buttons. Thus the user does not have to learn any coding language in order to interact with the system. This feature differentiate L'Algebrista from other computer environments quite popular in school practice such us LOGO and PASCAL, where the user must learn a peculiar language to interact with the computer.
- L'Algebrista is not able to do any transformation if it is not guided explicitly by the user using the above mentioned buttons. Differently from what happens with other symbolic manipulators, the user has the total control on the transformation activity.
- When proved, a new theorem may be represented as a new button and added to the system of axioms and theorems. Thus the evolution of the microworld will go on in parallel with the evolution of the theoretical system.

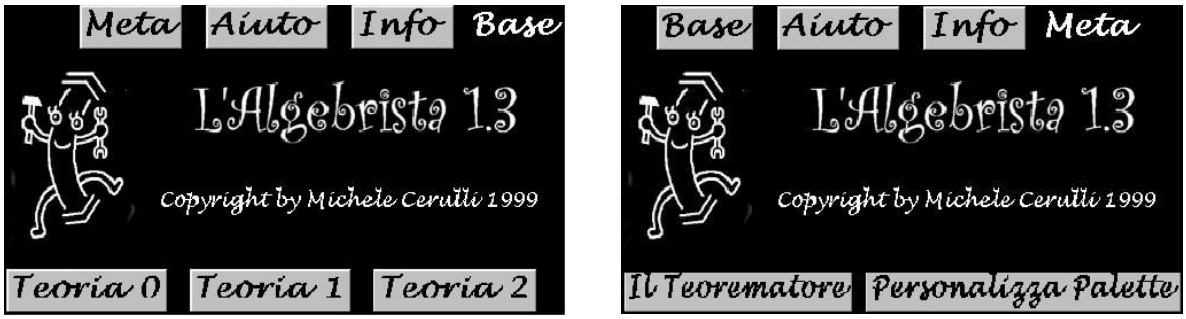

Fig1. - The *Base* menu and the *Meta* menu

3.1 Brief description of the software

After the start up sequence L'Algebrista offers the user to chose between four different menus: ***Base, Meta, Aiuto***[3] and ***Info***. As one might imagine, the menus *Info* and *Aiuto* give information concerning L'Algebrista and how to use its microworlds and environments. The *Info* menu

[3] "Aiuto" in Italian means "help". For the moment we only have an Italian version of the software.

contains the common explanations concerning the copyright and licence; the *Aiuto* menu contains information about the main features of the different menus and an "on-line help".

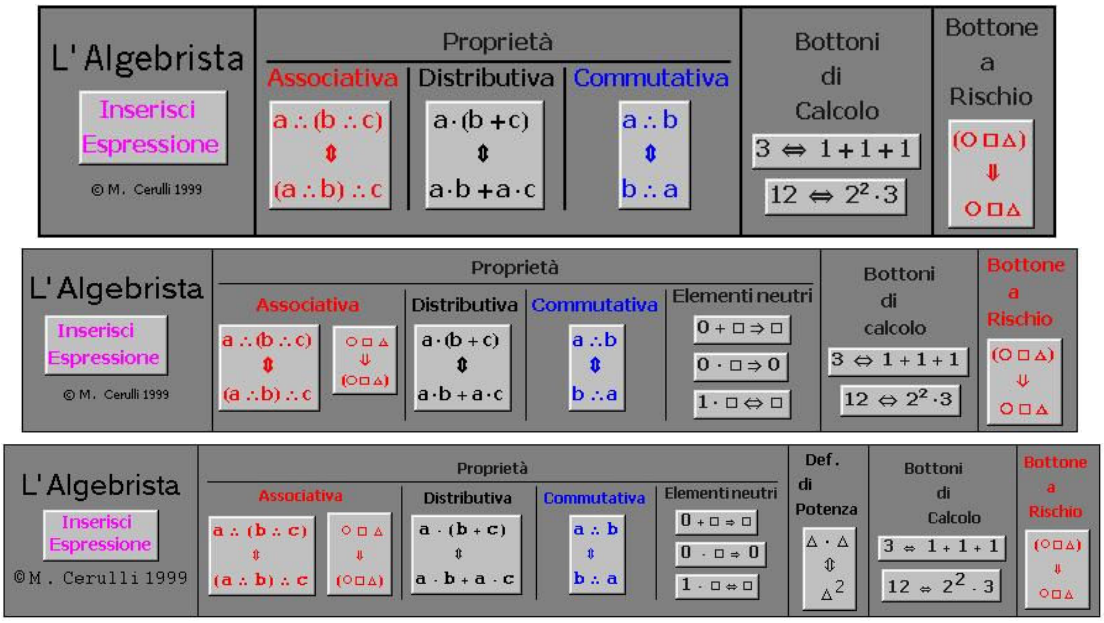

Fig.2 – Teoria 0, Teoria 1 and Teoria 2

3.2 The *Base* menu

This menu introduces the user to the main working environment of L'Algebrista; here the user can chose between several *Teorie* (Theories), i.e. microworlds of algebraic expressions. In fig. 1 it is possible to see the three basic theories we are using in a classroom experimentation which is going on by now.

Each *theory* is made of palettes (windows containing buttons) and notebooks (working environments). To start the activity the user has to write an expression in a notebook and insert it the microworld, then the manipulation of the expression will be carried out by selecting sub-expressions and clicking on the buttons available in the palettes. Every button (except one, as discussed below), always produces expressions which are equivalent to the expressions it is transforming.

3.3 The *Meta* menu

The word *meta* in this case stands for "meta theory", in fact this menu offers two instruments to be used to create new theories[4].

The first instrument is called ***Il Teorematore***, it lets the user create new buttons to represent new transformation rules that can be included in the palettes and used to manipulate expressions in L'Algebrista.

The second instrument is called *Personalizza Palette* (Palette personalisation) and is just a notebook containing a collection of ready made buttons and instructions concerning how to create a palette using those buttons and the buttons created with *Il Teorematore*.

With this two instruments teachers and pupils can actually build palettes including the axioms, definitions and theorems they prefer.

3.4 Description of the interaction with the basic microworld offered by L'Algebrista

Let's now describe the main commands of the theories presented in the *Base* menu analysing some peculiar aspects of the computer-user interaction.

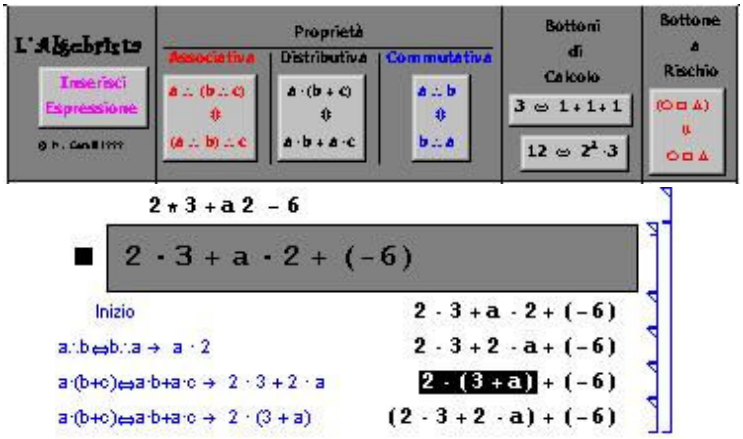

Fig 3. – In a notebook the user writes the expression to work with («2 * 3 + a 2 - 6» in the example), then after selecting it the button *Inserisci Espressione* is clicked, thus L'Algebrista creates a new working area where the buttons are active.

Fig.3 shows a palette of L'Algebrista; more precisely it represents the first theory we used in our teaching experiments and corresponds to *Teoria 0* in the *Base* menu. This palette is divided in four sectors corresponding to: the button *Inserisci Espressione (Insert expression)*; the buttons of the properties of sum and multiplication; the computations' buttons; the *risky button* ("*Bottone a*

[4] Recall that with *theory* we mean set of axioms, definition and theorems represented by buttons.

Rischio"). This partition is coherent with the distinction between the roles played by each button in classroom activities. In particular, the bottoms representing the properties of the operations were separated from the buttons that execute computations, in order to distinguish the activities of transformation based of the axioms, from those based on numerical computations.

An example of interaction is reproduced in fig. 3 . The user writes on a notebook the expression he/she wants to work with ("2*3+a 2 - 6" in our example), then he/she selects the expression and clicks *Inserisci Espressione* ("Insert Expression"), thus L'Algebrista creates a new working environment where the original expression is marked on its left with the label *Inizio* ("start"). The operation of *inserting* the expression is fundamental because it proclaims the entrance into the microworld where it is possible to act only using the buttons offered by L'Algebrista.

We observe that when an expression is *inserted*, its new instance comes out with some changes: every multiplication is represented with a dot ("•"), so either stars ("2*3") or spaces ("a 2") are substituted with a dot ("2•3+a •2"); every subtraction is transformed into sum and every division is transformed into multiplication. L'Algebrista does not know subtraction, and division: this follows from a precise didactical choice because we wanted pupils to work in a "commutative environment".

Interaction always happens by selecting a part of an expression and clicking on a button. The selection was designed so that it is not possible to select parts of expressions which are not sub-expressions from an algebraic point of view. For instance, given the expression **a•b+c** it is not possible to select **b+c**, if one tries to do it the software will automatically extend the selection to **a•b+c**; on the other hand one can select **a•b** or **c** or **a** etc. This feature corresponds to fact that the expressions of this microworld incorporate a fundamental algebraic characteristic of mathematical expressions: their tree structure.

Going back to the previous example, the expression can be now transformed by selecting the term **a•2** and clicking the 'commutative property button'; a new expression is produced (written just below), the term **2•a** is substituted by the term **a•2**, while on the left a label indicates the button used and the sub-expression it was applied on. Going forward we transformed one part of the expression using the distributive property, and in the following step, using the same button[5] we inverted the previous transformation. Coherently with our didactical hypothesis, the buttons incorporate all the functions of the properties of operations without advantaging any peculiar

direction. Note that most of the symbolic manipulator use another, different, command in order to invert a specific command.

L'Algebrista's buttons always produce a correct expression, that is equivalent to the original expressions to which they had been applied; the only exception is the *Risky Button* which is used to delete parenthesis: for instance it can transform **a+(b+c)** into **a+b+c** but it can also transform **a•(b+c)** into **a•b+c**. This button has been put aside and highlighted, so that the user can distinguish it from the others and use it with particular attention. Its meaning and its use is to be negotiated in the class in order to make clear the conventional use of parenthesis and its relation to algebraic axioms.

We conclude this section with a couple of observations on the notations used: the commutative and the associative properties have been represented using the symbol "∴", instead of "+" and "•", this is certainly related to a matter of economy, but also it is intended to familiarise students with generalisation of structures.

3.5 Il Teorematore

Il Teorematore is a peculiar environment which allows the user to create new buttons, fig.4 it shows the environment and its instructions.

The use of *Il Teorematore* is very simple, one just has to write the new transformation rule, to select it, and finally to click on the button *Teorema*. In our opinion it is fundamental that the user does not have to learn any coding language to create new buttons, but he/she just has to use mathematical symbols.

Fig.4 - Il Teorematore

Thanks to *Il Teorematore* the theory incorporated in L'Algebrista can grow together with the user's mathematical knowledge. In other words, the user can create as many buttons as he/she wants, and can then use them in his/her future interactions with L'Algebrista.

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[5] The button checks the structure of the expression and then decides how to transform it; in case no structure is recognised then the expression is left as it is.

Coherently with its basic principles, L'Algebrista, thanks to Il Teorematore, lets the user create new buttons corresponding to bi-directional transformations, that is buttons which incorporate all the functionality of equivalence relationships. This feature differentiate strongly L'Algebrista from most popular symbolic manipulators. In particular DERIVE does not allow the user to create any new command, while other didactical softwares (such as MILO and Theorist) let the user create only one direction commands that can be inverted only by using other commands.

Finally we observe that Il Teorematore **does not** check mathematical correctness of a new transformation rule. This is a consequence of a specific didactical choice: we want the pupil to be responsible for the validation of a new theorem or transformation rule. Thus it is the student who will have to control his/her set of axioms and theorems, i.e. the theory he/she is building in L'Algebrista.

3.6 Technical notes

L'Algebrista, is an application of the more popular software *Mathematica.* In order to run it needs Mathematica 3.0 (or more advanced versions), thus the hardware and software resources needed are the same needed to run *Mathematica*. In particular L'Algebrista is platform independent, within the range of the most popular Operative Systems (e.g. Linux, Macintosh, Windows).

The code is based on Mathematica language and the application consists of a set of notebooks and palettes.

Each mathematical expression is represented in two ways: "externally", following the usual mathematical notation, that is infix notation; "internally", using prefix notation.

Each button is coded using *Mathematica* graphical features and includes a function; such function translates the selected expression from infix to prefix notation, transforms the expression using a "transformation rule", and finally translates the new expression into infix notation.

The function corresponding to a transformation rule does not execute any computation on polynomials, actually the transformation of expressions is based simply on changes of structures. In other words, such functions extract the "leaf terms" from the tree structure of the expression and combine them in a new tree structure. This method is not optimized in terms of execution speed, but its main principle is what made simple the development of an instrument such as "Il

Teorematore", in fact in this way, the building of new transformation functions results straightforward.

4 Outlines of the classroom experimentation

A first experimentation was, carried out during the school year 1998/1999 in a 9th grade class, permitted to realise a final version of the prototype, which is now experimented in another class at the same level. Here we are not going to give a detailed description of the two experiments, but we will indicate the outlines of the sequence of activities and the basic ideas inspiring it.

First of all we recall that our educational goal concerns:

- to introduce pupils to symbolic manipulation;
- to introduce pupils to a theoretical perspective .

According to our hypothesis, the concept of *equivalence relation* is the basic principle underling symbolic manipulation, thus it represents the starting point of pupils activities.

We introduce the problem of comparing expressions, taking into account the fact that, at this school level, pupils consider numerical expressions as equivalent when they give the same number as result. Thus it is not difficult to negotiate the interpretation of numerical expressions as computation schemes, which will be equivalent if they give the same result.

The idea of interpreting expressions as computation schemes allows one to introduce the properties of sum and multiplication as principles (theory axioms) that determine a priori whether two computation schemes lead to the same result: if two expressions are equivalent on the base of such properties then the computation of the two expressions must lead to the same number. Thus a new equivalence relationship between expressions is introduced:

> *if one expression can be transformed into another using the properties of sum and multiplication, then the two expressions are equivalent*.

In the microworld L'Algebrista this corresponds to:

> *two expressions are equivalent if it is possible to transform one into the other using the given buttons*.

Once this equivalence relation is accepted, pupils are asked to compare expressions. A new terminology is introduced: one says that the equivalence of two expressions is **proved** if one expression is transformed into the other using the axioms; vice versa one says that the equivalence is **verified** if the calculation of both the expressions leads to the same result.

With literal expressions the difference between *proof* and *verification* becomes even more definite: the use of axioms becomes the only way to state the equivalence between two expressions, whilst numerical verification (substituting the letters with numbers and computing the expressions) becomes the main way to prove that two expressions are not equivalent.

Il Teorematore can then be used to add a selected choice of proven equivalencies to the set of buttons to be used for new proofs.

5 Semiotic Mediation

Within the Vygotskyan framework of semiotic mediation theory, a central role is played by the signs used to mediate mathematical meanings. Thus when analysing, and/or designing, an environment, its fundamental to study the nature of the signs to be used to mediate mathematical meanings. In our opinion, analysing didactic softwares should take into account such differences of representation so as their implications in terms of meaning construction and their functioning as instruments of semiotic mediation (Mariotti, in press).

A full discussion on this topic is beyond the scopes of this paper, thus we will limit to discuss on some aspects strictly related to L'Algebrista and more generally to symbolic manipulators.

Let us take the case of algebraic expressions as represented in L'Algebrista and algebraic expressions as represented in the paper and pencil environment. If we consider an expression, written on a sheet of paper, no interaction with the specific environment will make explicit its structure; in that case the structure of the expression can be implicitly imposed by the expert, but might be totally absent for novices.

The representation of an expression in L'Algebrista incorporates its mathematical tree structure[6], and this structure becomes explicit, “tangible”, when the user interacts with the environment[7]. This feature actually differentiates L'Algebrista also from other symbolic manipulators, such as DERIVE, where, in order to transform a sub-expression of a given expression, the user has to write it again in the buffer; in this case the user needs to know a priori (independently from the software environment) the structure of the expression, in order to chose the right sub-expression to rewrite. On the other hand, in the case of L’Algebrista, part of the theoretical control is incorporated in the computational object so that actions on the expression are submitted to that control. The specific *selection function*, which embeds such control, constitutes an external sign

[6] Actually this feature is inherited from *Mathematica*, and is called *structured selection*.
[7] See the description of the software above.

(Vygotsky, 1978) of the theoretical control, and as such it can function as a semiotic mediator. An example of how pupils use the selection sign as an external control referring to the theoretical properties, during the manipulation of an expression, is provided by the protocol in fig.5.

Expressions are not the only things that can be represented in a symbolic manipulator environment: actually any command represents some specific procedure or theorem. Again one can compare various softwares discussing, for example, how the distributive property (an axiom) is represented. Most symbolic manipulators (ex. *Mathematica*, DERIVE, MILO, Theorist) split the distributive property into two distinct commands, usually called **Factor** ("Multiply out" in MILO) and **Expand**. Thus, if the distributive property is expressed by the equality **a•(b+c)=a•b+a•c**, the command **Expand** corresponds to "going from left to right" and **Factor** corresponds to "going from right to left". **Factor** and **Expand** maybe considered one the inverse[8] of the other, and correspond to the two functions of the distributive property. In other terms Derive introduces a procedural interpretation of the sign "=", and may be an obstacle to the correct interpretation in relational terms.

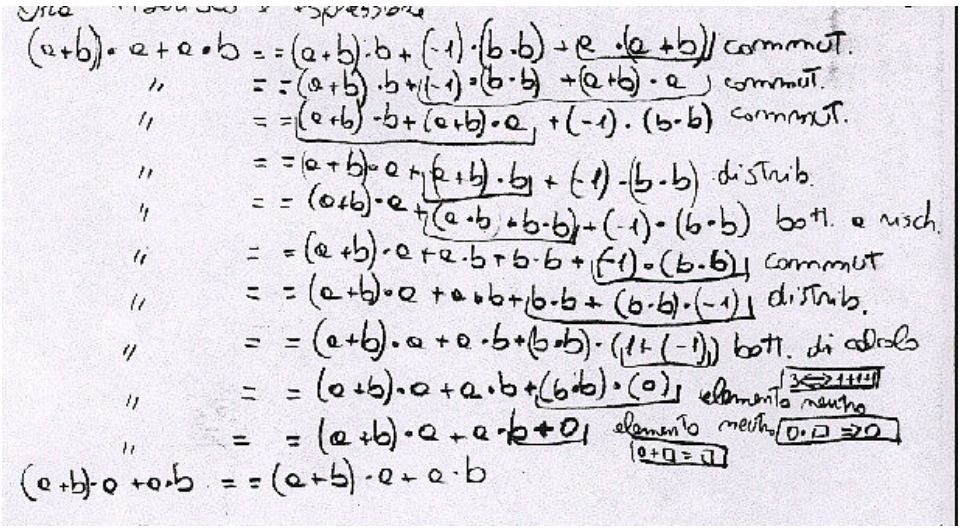

Fig.5 – In the case of a comparison task, performed in paper and pencil environment, the protocol shows that pupils use signs clearly derived from L'Algebrista, in particular the *selectin function*, or the iconography of the buttons.

In L'Algebrista the distributive property is represented by a unique button, called "button of distributive property", and represented as **a•(b+c)=a•b+a•c**. This button performs two

[8] Actually, for instance in the case of DERIVE, Factor and Expand perform a complex of transformations which cannot be related to one single axiom and because of their complexity, they cannot be considered one the inverse of the other.

transformations, one inverse of the other, so it actually *works* as an equivalence relationship: it allows the user to transform an expression into another and vice versa.

Furthermore the used iconography expresses the effect of the button on the algebraic structure of the expressions, whilst in the case of "Factor" and "Expand", this words do not say anything a priori about the relation between the structures of the two expressions involved.

In conclusion, the "button of distributive property" expresses the character of equivalence relationship of the distributive property, and incorporates both its functions. On the other hand, the commands Factor and Expand, represent the functions of the distributive property but cannot express the fact that it is an equivalence relationship.

Finally it is worth to observe how the L'Algebrista environment can offer instruments of semiotic mediation related to the theoretic aspects of algebra:

- expressions in L'Algebrista environments are signs of algebraic expressions;
- given buttons are signs of axioms and definitions;
- new buttons, built using Il Teorematore, are signs of theorems;
- transforming an expression into an other using the buttons corresponds to proving that the two expressions are equivalent, the produced chain of justified steps[9] corresponds to a proof.

The aim of the discussion, concerning the above mentioned examples, is to show that a very wide range of mathematical concepts can find a counterpart in computer environments. Furthermore, a single mathematical concept can be represented using very different signs that make explicit different aspects of the same concept. But the crucial point remains the effectiveness of such signs as instruments of semiotic mediation. The experimentation in progress aims at analysing the process of semiotic mediation as it is accomplished in the classroom practice.

6 Conclusions

The development of information technologies leads to many discussions, one of those concerns the revision of school curricula taking into account the changes brought by this development. The ideas we presented in this paper give an example of a new way to approach symbolic manipulation (Ita. "calcolo letterale"). Our proposal is to be considered as concerning questions relative to the introduction of pupils to theoretic thinking. Thus symbolic manipulation has been

[9] The justification of each step is reported on its left.

interpreted taking a theoretic perspective and the particular software environment has been designed as embedding Algebra Theory.

The axioms incorporated in the buttons of L'Algebrista becomes tools that pupils can learn to use to transform expressions in order to attain activities' goals. The distinction between buttons representing axioms, and buttons for computations, helps distinguishing the terms "proof" and "verification"; and may contribute to build the meaning of *proof* as well as the idea of *theory*. Furthermore the possibility of creating new theorems and making them usable, offered by "Il Teorematore", lets the student participate in the activity of *theory* evolution.

A research study has been set up with the aim of analysing the process of semiotic mediation in the case of L'Algebrista. The research project has been set up taking advantage of the previous study concerning the use of Cabri, the experimentation, still in progress, is devoted to analyse the joint use of L'Algebrista and Cabri: the main hypothesis concern the synergy provided by the two environments in rapport to the objective of developing a general idea of theory.

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