Geometry: dynamic intuition and theory

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Summary of the talk

Geometry is a school subject, but also and primarily geometry is a mathematical domain. As mathematical educators we are interested in geometry from both perspectives, as well as we are interested in the relationship between them.

First of all, geometry is a formalized theory, within the broader frame of mathematical theories. Actually, geometry pervades (permeates) a large part of mathematics, even some of its very recent developments, influencing and affecting both their origin and their evolutions.

Geometry as a theory of space offers a good model to control basic intuition in different and various mathematical domains.

A privileged relationship between elementary Geometry and physical reality must be recognized. According to the theory of ***figural concepts*** geometrical concepts are

"A mixture of two independent, defined entities that is abstract ideas (concepts), on one hand, and sensory representations reflecting some concrete operations, on the other"

(Fischbein,1993, pag. 140).

The question arises about the congruence between spatial cognition and abstract mathematical space, i.e. Geometry. Complete congruence between the two systems is not always assured. Far from being natural the move from intuition to geometry presents great difficulties . Two objectives can be stated for geometrical education.

- the necessity of developing a flexible interaction between images and concepts.
- the development of complex conceptual schemes, controlling the meanings, the relationships and the properties of a geometrical figure.

Dynamic Geometry a new contribution to an old issue.

A DGS[1], such as Cabri introduces a particular kind of images: screen images produced in a microworld. The main characteristic is their dynamic property: pictures can be dragged and they change under the effect of dragging. Different types of dragging modes are available, but each of them has the common feature of preserving some of the relationships characteristic of the original figure, in particular those relationships which defined its construction.

A number of studies have largely provided evidence supporting the general claim on the role of DGS in enhancing geometry teaching and learning (Goldengerg & Cuoco, 1998; Laborde, 1993; Healy, 1999) and I do not intend to present just a new example. What I would like to do is to focus on the basic features of the Cabri environment and discuss its relevance from a general point of view.

[1] The following remark concerns types of software which share with Cabri the general feature of a 'drag mode'; I mean for instance Sketchpad or Geometric Supposer.

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Taking a vygotskian perspective I want to discuss on the contribute of Cabri, and in particular of its 'tools', in the construction of mathematical meanings.

Two examples will be discussed in order to show the contribution of a DGS environment to the development of the key-ideas and more generally to geometrical intuition.

The first example is related to the idea of construction which is central in the tradition of elementary geometry and crucial to distinguish between theoretical and practical problems.

The second example is related to the idea of graph, perhaps the most familiar example of geometrical idea reinvested within the algebra domain.

Theoretical framework: Tools, signs and meaning

As said above, DGS offer a powerful environment incorporating the semantic domain of space and time, where the key notions of geometry can be grounded. This general idea can be interpreted in a Vygotskian perspective, according to the notion of semiotic mediation.

Vygotsky distinguishes between the function of mediation of *technical tools* and that of *psychological tools* (or *signs* or *tools of semiotic mediation*) (Vygotsky, 1978: 53). The use of the term *psychological tools*, that refers to signs as internally oriented, is based on the analogy between tools and signs, but also on the relationship that links specific tools and their externally oriented (for the mastering of nature) use to their internal counterpart (for the control of oneself) (ibid.: 55).Through the complex **process of internalization**, a tool may become a « psychological tool » and will shape new meanings; in this respect a tool may function as a semiotic mediator.

As far as the DGS Cabri is concerned, previous studies (Mariotti & Bartolini, 1998; Arzarello, 2000, Mariotti 2001a), 2001b), 2002,) have focused on the analysis of the specific elements of the microworld (dragging facility, commands available, macro …) as instruments of semiotic mediation that the teacher can use in order to introduce pupils to mathematical ideas. The theoretical hypothesis assumed is the following: a set of particular tools, among those available in the microworld Cabri may be considered as an external sign, referring to specific mathematical meanings, and as such they may become instruments of semiotic mediation, as long they are used by the teacher in the concrete realisation of classroom activity to introduce pupils to mathematics.

Basic pedagogical assumptions

Taking into account the correspondence between some Cabri tools and the meanings relevant to the didactic goals, a sequence of activities are designed and implemented in the classes, according to on some general assumptions about the teaching/learning processes:

- Tools are part of the construction process of meanings, then they can be used by the teacher to foster this process according to intended meanings
- Learning is both an individual and a social construction.

As a consequence the general structure of the activities consisted of two stages:

- firstly students were faced with tasks to be carried out in the Cabri environment,
- the various solutions were then discussed collectively under the guidance of the teacher. In this respect, Mathematical discussion as introduced by Bartolini Bussi (1998), constitutes a basic activity.

First example:The construction task

Since antiquity geometrical constructions have had a fundamental theoretical importance (Heath, 1956, p. 124) clearly illustrated by the history of the classic impossible problems, which so much puzzled the Greek geometers.

Actually, the theoretical meaning of geometrical constructions, i.e. e. the relationship between a geometrical construction and the theorem which validates it, is very complex and certainly, not immediate for students. It seems that the very nature of the construction problem makes it difficult to take a theoretical perspective, as clearly described and discussed by Schoenfeld (1985), and in a completely different school context by Mariotti (1996).

In spite of the long tradition, geometrical constructions have lost their position in the geometry curriculum, but the appearance of DGS has renewed the interest for constructions and the basic role played by construction has been brought on the scene by the instrumental approach related to the use of graphic tools.

In particular, Cabri-géomètre offers a microworld which embodies Euclidean Geometry, referring to the classic world of "ruler and compass" constructions. In fact, any Cabri-figure is the result of a construction process, i.e. it is obtained after the repeated use of tools, chosen among those available in the "tool bar"; moreover, the effect of most of the Cabri tools corresponds to the effect of the classic geometric tools, i.e. ruler and compass: a Cabri-figure is obtained intersecting lines, lines and circles, constructing perpendicular or parallel lines and the like.

But there is something more. The Cabri environment introduces a specific criterion of validation for the solution of the construction problems: a solution is valid if and only if the properties characterising the geometrical figure are *invariant* under the dragging test.

Thus, the dynamic system of Cabri-figures embodies a system of relationships consistent in the broad system of a geometrical theory; in other terms solving construction problems means accepting not only all the facilities of the software, but also accepting a logic system within which to make sense of them.

For all these reasons, the idea of *geometrical construction within the Cabri environment* may be considered as a *key to accessing* theoretical thinking and, according to the vygotskian perspective, we made the basic assumption that both the **Cabri commands** and the **Dragging** mode may function as a "tool of semiotic mediation" . In other terms, the dragging mode can be considered an external sing referring to theoretical validation, that is the dragging mode may become a tool of semiotic mediation to be used by the teacher to introduce pupils to a theoretical perspective.

Based on this assumption a long term experimental project was carried out in several classes, at the 9^{th} and 10^{th} grades level. Details on this experiments can be found in (Mariotti 2001a, 2001b).

According to the basic assumption of the relationship between Cabri constructions and geometrical theorems, working in the Cabri environment, under the guidance of the teacher, pupils were introduced to the geometrical theory.

Some example will be presented, in particular some excerpts from pupils productions, showing how pupils develop the meaning of theorem. More generally, I will discuss how pupils develop geometrical reasoning, as related to theoretical control on images.

Second example: function and graph

Function as change

Although not expressed in the classic mathematical definition of function, the idea of variation and co-variation is a crucial component of the notion of function, as Tall clearly states: "One purpose of the function is to present how things change" (Tall, 1996. p. 288). Our assumption is that grasping the idea of function requires grasping the idea of co-variation, i.e. e. conditional change. As cognitive analysis highlights, motion – space changing over time – can be considered as one of the basic primitive perceptions of “dynamic and continuous” (by using the terms of Malik, 1980) variation.

A connection to the basic metaphor should be preserved in the idea of graph, that is in the spatial representation of a function in a coordinate plane; however, this can be done by considering the graph as the trajectory of a moving point P (Laborde 1999, p.170), representing the dependent variable, according to the variation of a variable point M on the axis of abscissas, representing the independent variable. This complex interpretation requires to reintroduce time and to consider the co-variation of P and M as a relation between two interrelated variations depending on time, what we call a dynamic interpretation of a graph.

Unfortunately, this dynamic interpretation is often neglected in the textbooks. In any case, a dynamic interpretation of a graph cannot be externally experienced and remains a sort of mental experiment, impossible to be shared. Unlike the paper and pencil environment, which cannot afford the representation of change through motion, the DGS environment can. In DGS environments the idea of variation is grounded in motion, so that it is possible to experience variation in the form of motion: points can be dragged on the screen and represent the basic variables. As a consequence, DGS environments incorporate and represent the idea of variation and that of functional dependency.

Thus DGS offer a powerful environment incorporating the semantic domain of space and time, where the notion of function can be grounded. We call that particular instance of function “Dynamic Geometrical Function”. This general idea can be interpreted in a Vygotskian perspective, according to the notion of semiotic mediation.

The notion of trajectory

Based on the previous assumptions a teaching experiment was designed and carried out at the 9th-10th grade level, in different classes both in Italy and in France (details can be found in Laborde & Mariotti, 2002;Falcade, 2002).

The introduction of the variation and co-variation is achieved through exploring the effect of Macro constructions; a first definition of function, domain and image is elaborated, based on the interpretation of a geometric situation in terms of function. "Dragging", "Trace tool" and “Macro tool” are the key elements on which semiotic mediation is based.

The sequence of the d activities evelopes in two phases:

1. Introduction of the variation and co-variation through exploring the effect of Macro constructions; a first definition of function, domain and image was given, based on the interpretation of a geometric situation in terms of function. "Dragging", "Trace tool" and “Macro tool” are the key elements on which semiotic mediation is based.
2. Introduction of the idea of graph, working on a text, drawn from the original work of Euler (Euler, 1743). According to his *method* (accurately described in his text) Euler constructs a representative of a numerical function within the Geometry domain.

Euler’s method is intrinsecally dynamic and consists in generating the **trajectory** of a point in the plane.

Examples drawn from this experimental project will be presented with the aim to discuss onthe functioning of the semiotic mediation process. The twofold conception of trajectory, global and punctual, clearly emerged, in relation to dragging and Trace tools. We can say that the internalization of the dragging and Trace tools contributes to develop both the global and the punctual aspects of the idea of trajectory, and this in connection to the introduction of the notions of domain and image of function. According to our hypothesis, the twofold conception of trajectory as object and sequence of points has been reinvested both in the interpretation of Euler's text and in its appropriation as a method to conceive dynamically a graph of function both as a set of points and as a curve.

Conclusions

As a final remark, we come back tho the initial discussion concerning geometry permeating mathematical thinking. The example of the graph of a function is perhapes the most evident and paradigmatic. Spatial and, more generally, dynamic intuition when developed within the geometrical domain provides a powerful way of thinking, assuring to images the logic control necessary to establish and maintain the efficient links between what is seen and what is known.

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